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Finite Boundary Interpolation by Univalent Functions

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1. INTRODUCTION

Let $D=\{z\in\mathbb{C}: |z|<1\}$, and suppose that $z_1, z_2, ..., z_n$ and $w_1, w_2, ..., w_n$ are two collections of distinct points on ∂D arranged in counterclockwise order. Let $z_k=e^{i\alpha_k}$ and $w_k=e^{i\beta_k}$, where $\alpha_1<\alpha_2<\cdots<\alpha_n<\alpha_1+2\pi$ and $\beta_1<\beta_2<\cdots<\beta_n<\beta_1+2\pi$. We are interested in functions f which are analytic and univalent in D and satisfy the boundary interpolation $f(z_k)=w_k$ for $k=1, 2, ..., n$.

In particular we prove the following theorem.

THEOREM 1. *There is a function f which is analytic and univalent in the union of D and a neighborhood of $\{z_1, z_2, ..., z_n\}$ and continuous on $\bar{D}$ such that $f(z_k)=w_k$ for $k=1, 2, ..., n$. Furthermore, $|f(z)|=1$ if $|z|=1$ and z is sufficiently near any of the points z_k.*

Theorem 1 is related to considerations in the recent paper [3], where the following theorem about simultaneous peaking and interpolation is proved.

THOEREM A. (Clunie, Hallenbeck, and MacGregor). *There is a function f that is analytic and univalent in $\bar{D}$ and satisfies $|f(z)|<1$ for $|z|\leqslant 1$ and $z\neq z_k$ $(k=1, 2, ..., n)$ and $f(z_k)=w_k$ for $k=1, 2, ..., n$.*

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The proof of Theorem A is rather long and in several places non-constructive. The main steps in the argument rely on the following ideas: a peaking result for polynomials [1, p. 101]; an interpolation result for finite Blaschke products [2]; a starlike mapping having suitable properties [3, Lemma 1]; and an application of the Riemann mapping theorem for a domain formed from a disk by adding "channels."

Theorem 1 can be used to give a somewhat simpler and more constructive proof of Therem A. The argument relies on the following reslt, which is cntained in [3, Sect. 3] and is a weakened version of Theorem A. The proof of this result is elementary and provides a step by step procedure for obtaining the function from the given points. This function is a composition of a finite number of functions which are power functions, exponentials, or Möbius transformations. The argument for Theorem 1 also relies of properties of explicit functions which map D onto the complement of spirals.

THEOREM B. *There is a function f that is analytic and univalent in $\bar{D}$ and satisfies $|f(z)|<1$ for $|z|\leqslant 1$ and $z\neq z_k$ $(k=1, 2, ..., n)$ and $|f(z_k)|=1$ for $k=1, 2, ..., n$.*

Our proof of Theorem A is as follows. Let g be a function given by Theorem B and let $\zeta_k=g(z_k)$ for $k=1, 2, ..., n$. Let h be a function given by Theorem 1 for the two collections of points, $\zeta_1, \zeta_2, ..., \zeta_n$ and $w_1, w_2, ..., w_n$. Then $f=h\circ g$ satisfies Theorem A.

We also prove the following similar results.

THEOREM 2. *Suppose $a<x_1<x_2<\cdots<x_n<b$ and $y_1<y_2<\cdots<y_n$. There is a real-valued polynomial p which is univalent in a domain containing $[a, b]$ such that $p(x_k)=y_k$ for $k=1, 2, ..., n$.*

THEOREM 3. *Let $R=\{\zeta\in\mathbb{C}:\operatorname{Re}\zeta>0\}$ and let S be a subset of R such that $\partial S\cap\partial R=\{\zeta_1, \zeta_2, ..., \zeta_n\}$. Given $\varepsilon>0$, there is a function f that is analytic and univalent in a neighborhood of $S\cup\{\zeta_1, \zeta_2, ..., \zeta_n\}$, and continuous in $\bar{S}$ such that $f(\zeta_k)=\zeta_k$ for $k=1, 2, ..., n$ and $f(S)\subset\{\zeta: 0<\operatorname{Re}\zeta<\varepsilon\}$.*

Stated briefly, Theorem 3 provides mappings which keep ζ_k fixed while "squashing" the set S toward the imaginary axis. We ask whether Theorem 3 can be improved to include the conclusion $f(S)\subset\{\zeta: m-\varepsilon<\operatorname{Im}\zeta<M+\varepsilon\}$, where $m=\min_k \operatorname{Im}\zeta_k$ and $M=\max_k \operatorname{Im}\zeta_k$. Such a result may have implications when combined with Theorem 2 or similar facts.

2. Proof of Theorem 1

The proof of Theorem 1 depends on the following construction of a map of D onto a set consisting of D less a finite number of slits.

Suppose that m is an integer, $m \geqslant 2$, and let $\zeta_k = e^{i\gamma_k}$ for $k = 1, 2, ..., m$, where $\gamma_1 < \gamma_2 < \cdots < \gamma_m < \gamma_1 + 2\pi$. Assume that $\zeta' = e^{i\gamma'}$, where $\gamma_{m-1} < \gamma' < \gamma_1 + 2\pi$. We will obtain a function h which in particular is analytic in D and at $\zeta_1, \zeta_2, ..., \zeta_{m-1}, \zeta'$ such that $h(\zeta_k) = \zeta_k$ for $k = 1, 2, ..., m-1$ and $h(\zeta') = \zeta_m$.

If $\zeta' = \zeta_m$ then the identity function serves for h. Otherwise, define γ'' by

$$\gamma'' = \begin{cases} \frac{1}{2}(\gamma_{m-1} + \gamma_m), & \text{if } \gamma' > \gamma_m \\ \frac{1}{2}(\gamma_m + \gamma_1 + 2\pi), & \text{if } \gamma' < \gamma_m. \end{cases} \tag{1}$$

This gives a point $\zeta'' = e^{i\gamma''}$ so that on the counterclockwise arc on ∂D from ζ_{m-1} to ζ_1, ζ_m is between ζ' and ζ''. For $|z| < 1$, let

$$p(z) = \frac{1}{m-1} \sum_{j=1}^{m-1} \frac{1 + \bar{\zeta}_j z}{1 - \bar{\zeta}_j z}, \tag{2}$$

and define α by $|\alpha| < \pi/2$ and

$$\tan \alpha = \frac{1}{m-1} \sum_{j=1}^{m-1} \cot\left(\frac{\gamma_j - \gamma''}{2}\right). \tag{3}$$

Also, let the function g be defined by the differential equation

$$\frac{zg'(z)}{g(z)} = e^{i\alpha}[(\cos\alpha)\, p(z) + i \sin\alpha] \tag{4}$$

and the conditions $g(0) = 0$ and $g'(0) = 1$. Since $\operatorname{Re} p(z) > 0$ for $|z| < 1$ and $p(0) = 1$, it follows from [5, *p.* 52] that g is α-spiral-like. The function g maps D one-to-one onto the plane slit along $m-1$ Jordan arcs (spirals) connected only at infinity.

Let $C_1, C_2, ..., C_{m-1}$ denote the circular arcs on ∂D which correspond to the individual slits comprising $\mathbb{C} \backslash g(D)$. On each arc C_j there is a unique point ξ_j mapping to the tip of the corresponding slit. Equations (2), (3), and (4) imply that $g'(e^{i\gamma''}) = 0$, and therefore $\xi_{m-1} = \xi''$.

Let $\sigma_j = g(\xi_j)$ for $j = 1, 2, ..., m-1$, and for $t > 0$ let G_t denote the subset of $\mathbb{C}$ defined by $w \neq \sigma_j \exp[-e^{i\alpha}s]$ for $0 < s < t$ and $j = 1, 2, ..., m-1$. Properties of α-spiral-like functions imply that $G_t \subset g(D)$. Thus the function g_t defined by $g_t(z) = g^{-1}[\{\exp(-e^{i\alpha}t)\}\, g(z)]$ for $t > 0$ is analytic in D. Also g_t maps D one-to-one onto a subset of D formed by removing $m-1$ Jordan slits joined to ∂D at the points $\xi_1, \xi_2, ..., \xi_{m-1}$. Each point z with

$|z|=1$ and $z \neq \zeta_j$ $(j=1, 2, ..., m-1)$ is mapped by $\{\exp[-e^{i\alpha}t]\}\, g$ onto another (finite) point on the spiral containing $g(z)$ or on the extension of that spiral toward the origin. Therefore, $g_t(\zeta_j)=\zeta_j$ for $j=1, 2, ..., m-1$. The function g_t is continuous in t for each z in $\bar{D}$, and if $|z| \leqslant 1$ and $z \neq \zeta_j$ $(j=1, 2, ..., m-1)$ then $g_t(z) \to 0$ as $t \to \infty$. This implies that $g_{t_0}(\zeta')=\zeta_m$ for some $t_0>0$. The function g_{t_0} is analytic in D and is continuous in $\bar{D}$. Also, $|g_{t_0}(z)|=1$ if $|z|=1$ and z is sufficiently near any of the points $\zeta_1, \zeta_2, ..., \zeta_{m-1}, \zeta'$. The reflection principle implies that g_{t_0} is analytic at $\zeta_1, \zeta_2, ..., \zeta_{m-1}, \zeta'$ and the reflection also shows that g_{t_0} is univalent in the union of D and a neighborhood of $\{\zeta_1, \zeta_2, ..., \zeta_{m-1}, \zeta'\}$.

This obtains $h=g_{t_0}$. Geometrically stated, for each t, g_t fixes $\zeta_1, \zeta_2, ..., \zeta_{m-1}$ as $g_t(\zeta')$ moves along ∂D (which monotone argument) until it reaches ζ_m for the value $t=t_0$. The slit at ζ'' effectively pulls ζ' toward ζ_m.

We now prove Theorem 1. Let $z_1, z_2, ..., z_n$ and $w_1, w_2, ..., w_n$ be as described in the Introduction. If $n=1$, the function f is obtained by a rotation. For $n=2$, first rotate D mapping z_1 to w_1. Let z_2' be the image of z_2 under this rotation. The constructon above with $m=2$, $\zeta_1=w_1$, $\zeta_2=w_2$, and $\zeta'=z_2'$ yields a function g such that the composition of the rotation with g gives a suitable function f.

Suppose that the theorem holds for $n=N$. We will show that it holds for $n=N+1$. Let $z_1, z_2, ..., z_{N+1}$ and $w_1, w_2, ..., w_{N+1}$ be the given sets of points. There is a function f_N satisfying the theorem for the sets of points $z_1, z_2, ..., z_N$ and $w_1, w_2, ..., w_N$. In particular this provides a suitable neighborhood A of $\{z_1, z_2, ..., z_N\}$. Also A contains a disk $\{z: |z-z_N|<\varepsilon\}$, for some $\varepsilon>0$, which does not contain $z_1, z_2, \;, z_{N-1}, z_{N+1}$ and such that $w_k \notin f_N[\{z: |z-z_N|<\varepsilon\}]$ for $1 \leqslant k \leqslant N+1$ and $k \neq N$. Let $z'_{N+1}=e^{i\alpha'}$ be a point in $\{z: |z-z_N|<\varepsilon\}$ with $\alpha_N<\alpha'<\alpha_{N+1}$ (where $z_N=e^{i\alpha_N}$ and $z_{N+1}=e^{i\alpha_{N+1}}$). Let $w'_{N+1}=f_N(z'_{N+1})$.

The earlier argument gives a function h_1 which is analytic and univalent in the union of D and a neighborhood of $\{w_1, w_2, ..., w_N, w'_{N+1}\}$ such that $h_1(w_k)=w_k$ for $k=1, 2, ..., N$ and $h_1(w'_{N+1})=w_{N+1}$. Since f_N is analytic and univalent in $D \cup A$, it also has these properties in $D \cup B$, where B is a smaller neighborhood consisting of open disks centered at $z_1, z_2, ..., z_N, z'_{N+1}$ so that f_N maps these disks into the neighborhood of $\{w_1, w_2, ..., w_N, w'_{N+1}\}$ above.

The earlier argument also gives a function h_2 which is analytic and univalent in the union of D and a neighborhood of $\{z_1, z_2, ..., z_{N+1}\}$ such that $h_2(z_k)=z_k$ for $k=1, 2, ..., N$ and $h_2(z_{N+1})=z'_{N+1}$. Some smaller neighborhood of $\{z_1, z_2, ..., z_{N+1}\}$ is mapped by h_2 into B.

The properties of h_1 and h_2 imply that $f=h_1 \circ f_N \circ h_2$ satisfies the conclusions of the theorem associated with the points $z_1, z_2, ..., z_{N+1}$ and $w_1, w_2, ..., w_{N+1}$.

3. Proof of Theorem 2

The proof of Theorem 2 depends on interpolation and approximation reslts about polynomials and is set up by the following lemmas.

Lemma 1. *Suppose* $a<b$, $c<d$, *and* $0<\varepsilon<(d-c)/(b-a)$. *There is a cubic polynomial* f *such that* $f(a)=c$, $f(b)=d$, $f'(a)=f'(b)=\varepsilon$ *and* $\min\{f'(x): a \leqslant x \leqslant b\}=\varepsilon$.

Proof. A translation of variables implies that there is no loss of generality to assume that $a=c=0$. Let $t>\varepsilon$ and let g be the quadratic polynomial such that $g(0)=\varepsilon$, $g(b/2)=t$, and $g(b)=\varepsilon$. Define f by $f(x)=\int_0^x g(s)\,ds$. Then $f(0)=0$, $f'(0)=f'(b)=\varepsilon$, and $f'(x)=g(x)\geqslant\varepsilon$ for $0\leqslant x\leqslant b$. Since $\int_0^b (g(s)-\varepsilon)\,ds \to 0$ as $t\to\varepsilon$, it follows that $f(b)\to\varepsilon b$ as $t\to\varepsilon$. Also, $f(b)\to\infty$ as $t\to\infty$. The condition $0<\varepsilon<d/b$ and the continuity of f assure that there is a value of t for which $f(b)=d$.

Lemma 2. *Suppose* $a<x_1<x_2<\cdots<x_n<b$ *and* $y_1<y_2<\cdots<y_n$. *There is a function* f *defined and continuously differentiable on* $[a, b]$ *such that* $f(x_k)=y_k$ *for* $k=1, 2, \ldots, n$ *and* $\min\{f'(x)\colon a\leqslant x\leqslant b\}>0$. (*Here and later, derivatives at end points are one-sided limits.*)

Proof. Choose y_0 and y_{n+1} such that $y_0<y_1$ and $y_{n+1}>y_n$ and let $x_0=a$ and $x_{n+1}=b$. Choose ε such that $0<\varepsilon<\min\{(y_{k+1}-y_k)/(x_{k+1}-x_k)\colon k=0, 1, \ldots, n\}$. Lemma 1 implies that there is a cubic polynomial in each of the intervals $[x_k, x_{k+1}]$ for $k=0, 1, \ldots, n$ which piecewise defines a function f on $[a, b]$ which is continuously differentiable. Also, $\min\{f'(x)\colon a\leqslant x\leqslant b\}=\varepsilon>0$.

Lemma 3. *Suppose* $a<x_1<x_2<\cdots<x_n<b$ *and* $y_1<y_2<\cdots<y_n$. *There is a polynomial* p *such that* $p(x_k)=y_k$ *for* $k=1, 2, \ldots, n$ *and* $\min\{p'(x)\colon a\leqslant x\leqslant b\}>0$.

Proof. Let f satisfy Lemma 2 and let $\varepsilon=\min\{f'(x): a\leqslant x\leqslant b\}$. Given $\delta>0$, then by [4, p. 113] there is a polynomial q such that

$$\max\{|f(x)-q(x)| : a\leqslant x\leqslant b\}<\delta \tag{5}$$

and

$$\max\{|f'(x)-q'(x)| : a\leqslant x\leqslant b\}<\delta. \tag{6}$$

Let r be the polynomial which interpolates the values $f(x_k)-q(x_k)$ for $k=1, 2, \ldots, n$. Then r can be expressed

$$r(x)=\sum_{k=1}^{n} [f(x_k)-q(x_k)]\, P_k(x), \tag{7}$$

where P_k is the polynomial of degree at most n such that $P_k(x_j)=0$ for $j \neq k$ and $P_k(x_k)=1$. Let $M_k = \max\{|P_k'(x)| : a \leqslant x \leqslant b\}$ and let $M=\sum_{k=1}^{n} M_k$. Equations (5) and (7) imply that $r'(x) \geqslant -\delta M$ for $a \leqslant x \leqslant b$.

The polynomial $p=r+q$ satisfies $p(x_k)=f(x_k)=y_k$ for $k=1, 2, ..., n$. If $a \leqslant x \leqslant b$ then (6) and the lower bound on r' imply $p'(x)=r'(x)+q'(x) > f'(x)-\delta-\delta M \geqslant \varepsilon - \delta - \delta M$. Therefore $\min\{p'(x)\colon\ a \leqslant x \leqslant b\} > 0$ for sufficient small δ.

Remark. No claim is made in Lemma 3 about the degree of p. In general, the Lagrange solution of the interpolation $p(x_k)=y_k$ with the conditions of Lemma 3 is not necessarily increasing on $[x_1, x_n]$. This suggests the problem of determining whether there are upper bounds on the degree of p which depend on n and/or the "spread" of the points x_k, y_k.

Proof of Theorem 2. Let p be a polynomial given by Lemma 3. We will show that there is a neighborhood (in the plane) of $[a, b]$ in which p is univalent. On the contrary, assume there is no such neighborhood. This implies there are two sequences $\{z_k\}$ and $\{z_k'\}$ with $z_k \neq z_k'$ and $p(z_k)=p(z_k')$ for $k=1, 2, ...,$ and each sequence has an accumulation point in $[a, b]$. Consideration of subsequences implies that we may assume that $z_k \to x_0$ and $z_k' \to x_0'$ with x_0 and x_0' in $[a, b]$. Thus $p(x_0)=p(x_0')$, and since p is strictly increasing on $[a, b]$, this requires $x_0 = x_0'$. However, $p'(x_0) \neq 0$, and therefore p is univalent in some neighborhood (in the plane) of x_0. This contradicts $p(z_k)=p(z_k')$ for sufficiently large k.

4. Proof of Theorem 3

We first note that Theorem 1 has an equivalent formulation for suitable domains which are conformally equivalent to D. For R, this is obtained by the introduction of a Möbius transformation and applies to two sets of n complex numbers on $\{\zeta : \operatorname{Re} \zeta = 0\}$ in the same conformal order.

In the case S is unbounded, first consider a mapping $\zeta \to 1/(\zeta - \zeta')$ which sends S to a bounded set T in R any complex number ζ' with $\operatorname{Re} \zeta' = 0$ and $\zeta' \neq \zeta_k$ for $k=1, 2, ..., n$. In particular, $T \subset \{\zeta : 0 < \operatorname{Re} \zeta < M\}$ for some M. Let $\zeta_k' = 1/(\zeta_k - \zeta')$. Theorem 1 implies that for each ρ with $0<\rho<1$ there is a function g_ρ which maps $\{\zeta : 0 < \operatorname{Re} \zeta < M\}$ into itself and is analytic at $\rho\zeta_k'$. Also, $g_\rho(\rho\zeta_k') = \zeta_k'$ for $k=1, 2, ..., n$ and $g_\rho(0)=0$. The function g_ρ with $\rho M < \varepsilon$ satisfies the conditions on f in the theorem. In the case S is bounded, the auxiliary mapping $\zeta \mapsto 1/(\zeta - \zeta')$ is not needed.

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